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The Study of the Solving Methods of the Rubik's Cube and the Analysis of Its Derived Algorithms

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Abstract. The major of the Rubik's Cube reduction algorithm is to construct an efficient operation sequence through mathematical tools (such as group theory). Different algorithms are based on the balance between the space division of the Rubik's Cube structure and the efficiency of operation, forming mainstream methods such as layer-first method, CFOP, ROUX, and bridge method. Starting from the underlying mathematical principles, this paper analyzes the design motivations and performance characteristics of different algorithms, and selects the CFOP algorithm for in-depth analysis.

Keywords: Commutator, conjugation, solving the Rubik's cube, algorithms comparison and analysis, Cayley graph

1. Introduction

Rubik's Cube is a three-dimensional puzzle toy invented by Hungarian architect Ern ő Rubik in 1974. The standard Rubik's Cube is a 3×3×3 cube structure with 6 faces, each covered with 9 small blocks, and colors are white, yellow, blue, green, red, and orange. The core structure of the Rubik's Cube includes the central block, which is fixed and determines the color of each face. Edge block (located on the edge, with two colors). Corner block: (located on the corner, with three colors.)

Rubik, Professor of architecture in Hungary, who invented the Rubik's Cube. The Rubik's international copyright issues. The original intention of Rubik's Cube was helping students to learn geometry more easily, interestingly, he spent roughly a month to solve the Cube that invented by him. It shows that even he invented the Rubik's Cube, however, but his original intention was not to create n educational toy. This is an episode about the history of the Rubik's Cube.

This paper aims to introduce the essential ideas behind solving the Rubik's cube, especially targeted to high school students similar to us as we realize that by understanding the Rubik's cube in a mathematical way may provide a spectacular insight to what does group theory does and a opportunity of learning its basic knowledge in a practical way.

1.1. Basic group theory definition

Definition 2. 1. 1 Group A group G is consists of a set of objects and a binary operator A group (G, ×) is a set G with a binary operation $\times : G \times G \to G$, satisfying:

- Associativity: $(a \times b) \times c = a \times (b \times c)(a \times b) \times c = a \times (b \times c)$ for all $a,\ b,\ c \in G$,

• Identity element: There exists $e \in G$ such that $e \times a = a \times e = a$ for all $a \in G$
• Inverse element: For each $a \in G$, there exists $a^{-1} \in G$ such that
• $a \times a^{-1} = a^{-1} \times a = e.$
There are some basic theorems about the groups:
1. The identity element, e, is unique.
2. If $a \times b = e$, then $a = b^{-1}$.
3. If $a \times x = b \times x$, then $a = b$.
4. The inverse of ab is $b^{-1}a^{-1}$.
Examples of group
Integers under addition: (Z, +) The set of all integers with the operation of addition (abelian group).
Real numbers under addition: (R, +) The set of real numbers with the operation of addition (abelian group).
Definition 2. 1. 2 Permutation: Let S be a set, then the permutation of S is a bijection $\sigma : S \to S$. In other words, the permutation of n letters is a one-to-one function
$\sigma : \{1, 2, 3, \ldots, n\} \to \{1, 2, 3, \ldots, n\}$, and the set of all functions is denoted S_n, the symmetry group.
With this definition we can now develop a useful notation to represent a specific permutation which belong to S_n.

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & \cdots & \mathrm{n} \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(\mathrm{n}) \end{bmatrix}$$

Note that the upper row seems to be quite unnecessary to be visibly shown, in this case we will simply the notation by concealing the upper row.

$$\sigma = [\sigma(1)\ \sigma(2)\ \sigma(3)\ \cdots\ \sigma(n)]$$

To further simply this, we will consider a more convenient notation in terms of the cycles in this permutation. Take

$$\sigma = [2\,5\,4\,3\,6\,7\,1\,8\,9]$$

for example, we start with 1, and 1 get mapped to 2 via σ, then 2 get mapped to 5, then 5 to 6, 6 to 7, and eventually 7 back to 1, we can represent this entire process by a 'cycle':

$$(1\,2\,5\,6\,7)$$

Through further observation, we can notice that (3 4) is also a cycle involved in σ, 8 and 9 each is a cycle of itself, which we called single cycle, denote (8) and (9) (In the future representation of permutation, we will conceal single cycle as it is even more convenient.) Therefore, the entire permutation could be expressed as:

$$(1\ 2\ 5\ 6\ 7)\ (3\ 4)\ (8)\ (9)$$

Note that the order of expressing each cycle will not matter, as for example $(1\ 2)\ (3\ 4)$ is completely identical to $(3\ 4)\ (1\ 2)$, as long as there no overlapped elements in each cycle, which this paper will not cover. Also, the order of the inside expression of each cycle will not matter as well, for example, $(1\ 2\ 5\ 6\ 7) = (2\ 5\ 6\ 7\ 1)$, as long as they represent the same transformation cycle.

Definition 2. 1. 3 Transposition: A transposition is a 2-cycle permutation

Definition 2. 1. 4 Order: In group theory, the order of a group G, denoted |G|, is defined as the cardinality (number of elements) of the set G. Formally:

$$\text{Order of } G = \left|G\right| = \begin{cases} \text{Number of distinct elements in } G & \text{if } G \text{ is finite,} \\ \infty & \text{if } G \text{ is infinite.} \end{cases}$$

Definition 2. 1. 5 Symmetric Group:

Let X be a non-empty set. The symmetric group on X, denoted S_n, is the set of all bijections (permutations) $\sigma: X \to X$ under the operation of function composition. Formally, S_n satisfies the group axioms:

- Closure: If $\sigma, \tau \in S_n$, then $\sigma \circ \tau \in S_n$.
- Associativity: $(\sigma \circ \tau) \circ \rho = \sigma \circ (\tau \circ \rho)$ for all $\sigma, \tau, \rho \in S_n$.
- Identity: The identity map $e_x: X \to X$, where $e_x(x) = x$ for all $x \in X$, satisfies $\sigma \circ e_x = \sigma = e_x \circ \sigma$.
- Inverses: For every $\sigma \in Sn$, there exists $\sigma^{-1} \in S^n$ such that $\sigma \circ \sigma^{-1} = e_x$.

When $X = \{1, 2, \ldots, n\}$, the group is denoted S_n and called the symmetric group of degree n. Its order is $|S_n| = n!$ (the number of permutations of n elements).

Key Properties

- S_n is non-abelian for $n \geq 3$.
- Permutations are often expressed in cycle notation (e.g. (1 2 3)) or as permutation matrices.

Definition 2. 1. 6 Alternating Group

Let S_n be the symmetric group on n symbols. The alternating group of degree n, denoted A_n, is the subgroup of S_n consisting of all even permutations (permutations that can be written as an even number of transpositions). Formally, A_n satisfies the group axioms under the composition of functions:

- Closure: The composition of two even permutations is an even permutation.
- Associativity: Inherited from S_n, composition is associative.
- Identity: The identity permutation id $\in S_n$ is even and serves as the identity element.
- Inverses: The inverse of an even permutation is also even.

The order of A_n is $|A_n| = \frac{n!}{2}$, since exactly half of the permutations in S_n are even.

Definition 2. 1. 7 Group Homomorphism: boxed Let (G, ×) and (H, ⋆) be groups. A function $\phi: G \to H$ is a group homomorphism if for all $a, b \in G$, $\phi(a \times b) = \phi(a) \star \phi(b)$

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Group homomorphism is like a "translation rule" between two groups (mathematical structures with operations): You can first operate the two elements in the first group, and then convert them in a different form (but the mathematical structure of the operation cannot be changed) to the second group, and we will get the same result.

For example: Real number addition group operation: 2 + 3 = 5 can be translated into real number exponential group: $e^2 \times e^3 = e^{2+3} = e^5$

1.2. The notation method of Rubik's cube and basic group theory knowledge related to Rubik's cube

The Rubik's cube is composed of 26 'cubies'. When working with Rubik's cube, it is always helpful to come up with a systematic method to name each cubie, instead of using different color to name each cubie, naming it in terms of their location may be more useful. In this thesis, we will continue to use David Singmaster's notation method, where we name each face by its first letter, hence we call them right (r), left (l), up (u), down (d), bottom (b), front (f).

When we consider the exact position of a specific corner or edge cubie, its orientation is also important, which is called 'oriented cubies'. However, in this thesis, we won't care which face is named first, that is, we are talking about 'unoriented cubies'. (For example, urf, rfu, and fur is identical.) Similarly, in order to name the edge and center cubies, we just name each individuals cubies by its visible faces. (such as u-f is an edge cubie and f is just a center cubie)

It is important to separate the concept of cubicles from cubies. The concept 'cubicles' are labeled the same way as cubies, but they are referring to the space where the initial cubies live in in the solved configuration. In other words, any moves will not change the cubicles position of the cube.

Now, we want to give name to every legal movement of the Rubik's cube, we use capital letters R, F, U, D, B, L to denote 90 degrees clockwise twists of the corresponding faces. And we use their lowercase to denote their reverse, which is a 90 degrees clockwise rotation, where for example, $R = r^{-1}$. In addition, if M is a move and C is a cubie, let M (C), denote the cubicle the cubie is standing on after the move M. (For example, RF (ur) = br)

It is quite intuitive that we can actually turn the set of moves into a group, denote (G, ×) And the elements of G will be all of the combinations of possible moves of Rubik's cube. And now we are going to prove this considering the four fundamental properties of a group:

1. G is certainly closed, since if M1 and M2 are two arbitrary moves, the combination of M1 and M2 are also the elements of the group.

2. There is an identity element in G, which is the move 'do nothing', since if any move combines with the mover 'do nothing' is still itself.

3. If M is a move, then we can definitely reverse the steps of the move to get M −1, which is basically the same as doing nothing.

4. Finally, in order to show the group is associative, this is not as easy as the previous three properties. To show this, we have to prove $(M1 \times M2) \times M3 = M1 \times (M2 \times M3)$, this is the same thing as to prove that both set of moves do the same thing to every cubie, which is $((M1 \times M2) \times M3)(C) = [M1 \times (M2 \times M3)](C)$.

Since $[(M1 \times M2) \times M3](C) = M3([(M1 \times M2)](C)) = M3(M2([M1](c)))$,

while $[M1 \times (M2 \times M3)](C) = (M2 \times M3)[(M1)(C)] = M3(M2([M1](c))$,

hence proved.

(Statement: Note that Rubik's cube is not commutative)

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1.3. The order of the Rubik's group

To find the total possible permutation of cubies in a Rubik's cube, we should separately consider edge and corner cubies since they are independent to each other (Notes that we should not consider the center cubies due to the symmetrical property of the Rubik's cube.) Firstly, we initially may want eight corner cubies to fit in eight cubicles, meaning that they are 8! positionings of corner cubies. If we also consider the orientation of the cubies, this will lead to 38possibilities for each positionings. Similar methodology could be done to the edge cubies as well, but with 12! permutations and 2^{12} possibilities for each permutation. Overall, this means we have in total of $2^{12}\times3^{8}\times8!\ \times12!$ possible arrangement of the cube.

Nevertheless, it is important to state that not all configurations are 'valid', meaning a position can reach through a set of legal moves from the starting configurations. Therefore, this number (about 5. 19×10^{20}) is not the actual order of the Rubik's group. In order to show this and to be

more specific, the number of configurations that are 'invalid', we need to consider the parity of the Rubik's group.

Every possible permutation of the Rubik's cube would involve a set of turns, which is obvious to see that every turn has an even parity with regard to the movements of cubies. Without loss of generality, consider a single move F, we could express this by using 24-cycles of edge and corner cubies:

$$F\ =\ (fl\ up\ fr\ df)\ (ufl\ urf\ drf\ dfl)$$

Each 4-cycles could be decomposed into three 2-cycles, which results in total of six 2-cycles, indicating an even parity per each move. This means the no combination of moves may change only two cubies around. (or any other odd permutations) However, not all even permutation could be achieved through legal combination of moves as well, since we should also consider the parity of the orientation of corner and edge cubies. In order to demonstrate the orientation of edge cubies clearly, a clever notation method is used in this thesis: Imagine a Rubik's cube that are held in space with its center cubies all constrained, let the center of the Rubik's cube also be the center of a set of three-dimensional coordinate axes so that every axis may pass through a particular center cubie. Each edge cubies will be assigned a direction that matched with the direction of paralleled axis passing through. Therefore, in the initial configuration, there will be 4 edge direction pointing in the same direction that aligns with its paralleled axis, and this is what happens after a face is rotated 90 degrees counter clockwise.

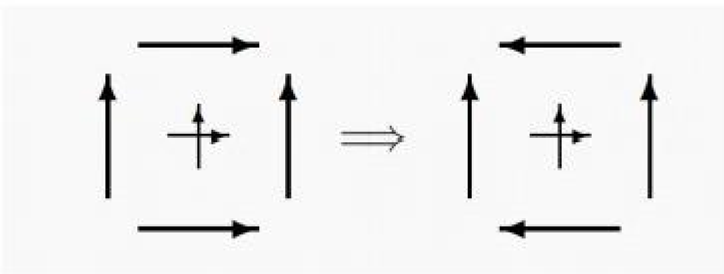

Figure 1. A demonstration of the notation method

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Without loss of generality, it is clear that exactly two of the arrows' direction has been flipped, this means that every turn will be involved an even flipped of edge cubies. Since we assumed that initially zero edge will be flipped, hence every possible configuration will be reached only if its edge cubies remain even parity.

We will also suggest a conclusion about the parity of corner cubies using slightly different method, and showing that the total rotation must be a multiple of 360 degree if it is a valid configuration. Put a mark on the up and down faces for every corner cubie, after a move such as R, it is clear that not all marks would still remain on the top or the bottom side. Consider for each cubicle, the mark will either remains constant, rotated 120 degrees or 240 degrees counter-clockwise, adding up all the rotations may draw to a simple pattern, that it has to be a multiple of 360. To be more specific, we can observe that after the move R two marks has been rotated 120 degrees and two marks has been rotated for 240 degrees, which add up to be 720 degrees in total, hence proved. This means if we now only consider the orientation of corner cubies, only one third of the assemblies put all the cubies in the right orientation, in other times it will be either lefts out by 120 or 240 degrees in total.

In conclusion, by considering the parity of the permutations of cubies, the orientation of edge cubies, and the orientation of corner cubies, we are now being able to paint a complete picture of the Rubik's cube by group homomorphism.

Proposition 2. 3. 1 (corner orientation invariant): In a solvable position, the sum of the corner orientations is 0 (mod 3) with respect to any frame of reference [3].

Proposition 2. 3. 2 (edge orientation invariant): In a solvable position, the sum of the edge orientations is 0 (mod 2) with respect to any frame of reference [3].

Since there are both half a chances of getting an even permutation of cubies and even number of edge cubies flipped, and a one out of third possibilities of having the correct orientation of the corner cubies, there are in total of only 1/12 amount of configurations that are valid in total, hence we are now being able to calculate the actual order of the Rubik's cube group, $2^{12}\times12!\times3^{8}\times8!\times2=$ 43252003274489856000. People may doubt the significance of even calculating this number; however, this number has actually played a crucial at least in the aspects of determining the lower limit of 'God number' which we will introduce it later on of this thesis.

The set of the configurations of a 3×3×3 Rubik's cube has size about $5.\ 19\times10^{12}$. It is impossible to deal directly with this huge set. The strategy is to break down these sets into some smaller individual sets with a certain feature and then to consider the methods to each set and compose them together. Thus, we can progressively restore the Rubik cube. For example, we can break the Rubik's cube into the set of corner cubes and the set of edge cubes. The reason is that the corner cube and the edge cube have a disjoint orbit. This is obvious since the physical structure implies that an edge will never go to a corner and vice versa. The difference between their way of motion implies that their position is inconvertible. In this way we can consider first restore the corner and then modify the position of edge by some operation that retains this symmetry (for instance, rotate the middle slice and do double flip). Similarly, other kinds of set division have different strategy and formula to approach the goal with distinct perspectives. However, to be realistic, the basic moves we can make is just rotate it's six faces-U, D, L, R, F, and B, there are always single moves that reacts on more than one set we expected exists and thereby leads to some undesirable effect on other blocks. To reduce this, we use commutators and conjugate sequence can apply some small adjustment over the original formula to reduce other block commutations as possible. Overall, we can apply that formula to reach our purpose.

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we have in total of $2^{12} \times 3^8 \times 8! \times 12!$ possible arrangement of the cube [1].

Nevertheless, it is important to state that not all configurations are 'valid', meaning a position can reach through a set of legal moves from the starting configurations. Therefore, this number (about 5. 19 × 10^{20}) is not the actual order of the Rubik's group. In order to show this and to be

the action is offset. For example, sail and sail back to the island could be seen as a simple reverse action that ultimately influences very few events. In these particular cases, overall, the only thing that has been altered is the position of treasure. This idea could be particularly beneficial in solving Rubik's cube, as eventually we want to alter the position of different cubies to solve the Rubik's cube while not influencing the original structure that we previously done. (Which we will further demonstrates later, this statement could be seen as an intuitive way of understanding the essence of conjugation and commutator.)

In mathematical terms, step 1 and 3 could be seen as a macro P and its inverse P^{-1}, (the concept macro refers to a set of moves that usually have a particular purpose such as alternating the position of 2 cubies). While M is usually a single move that basically twists the faces you want to operate P on, P and P^{-1} aim to restore the cubies that are not affected by M. The result PMP^{-1} is called 'The conjugation of M by P'.

For example, suppose the macro you know is "Flip UF, UL"" which we will repeatedly discuss later, but to solve the particular jumbled cube you're holding, you need to flip edge cubies opposite to the previous cubies on the bottom. You would turn the cube over and then do two twists to put the cubies that need flipping in the front-up and left-up positions, apply the macro, and then move the cube as a whole back to its original direction. Therefore, most conjugation-related macro could be seen as 'the change of coordinates' of the cube, which generalize the usefulness of a macro by using the symmetrical property of the Rubik's cube.

If A and B are two operations or elements of the Rubik's cube group, then we can now define the commutator of A and B, denotes $[A, B] = ABA^{-1}B^{-1}$, which have basically the same idea of conjugation as well. In a commutative system, it is quite apparent that [A, B] = e, as we can freely switching the sequence of operations which results in, they all canceled out. However, this is certainly not the case for G, the Rubik's cube group. In this case, the commutator could be seen as a detector of the commutative property of a group. If we think even deeper, we can realize that for that group which does not commutate, the commutators could be also seen as a sort of 'measurement of how commutative they are'. (However, since this is not the major topic of our thesis, we will not further discuss this idea.)

It could be concluded that although the expression of commutator and conjugation looks similar, they essentially perform entirely different functions, where conjugation generally functions in adapting a macro to a new location, while commutator is used to create a new macro with minimal effects.

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In order to showcase the functions of commutator in solving the Rubik's cube, here is a simple example, let $a = (1,\ 2,\ 3,\ 4,\ 5,\ 6)\,(7,\ 8,\ 9)\,(10,\ 11,\ 12,\ 13,\ 14)$, and let $b = (7,9)(15,16,17,18,19,20)$, it is not hard to calculate [a, b] = (7, 9, 8). This result is quite counter-intuitive since both a and b are complicated permutations but their commutator is quite 'simple'. Nevertheless, this phenomenon has an easy explanation, although a and b both moves a lot of objects, the only common cycles are 7, 8, and 9, while other permutation cycle that has not been altered by other permutation will be offset by its own inverse. For instance, consider the cycle of (1, 2, 3, 4, 5, 6) in a, it moves each number one step further and 6 is transverse back to 1, since b does not influence any number in the cycle, it will be eventually restore by a^{-1}, hence we can say that these two permutation is 'almost commutative' since they only alter the positions of very few objects. However, it is important to suggests that the statement that 'two permutations is almost commute when they move only few objects in common' is not very rigorous, which can be, at times, entirely untrue. We just aim to demonstrate the idea behind commutators, that is, functions to undo lots of 'irrelevant moves' while changing the necessary cubies when the operation is 'almost commutative' (Disregarding of the 'complexity' of the macros, referring to basically the number of cycles a permutation is consisted of)

Now, we will discuss one of the practical uses of the concept commutators in order to build up a very useful macro: the 'Flip UF, UL' macro (without changing other cubies' position) Here's the basic strategy, suppose the existence of a combination of moves, denote M, which functions in flipping only one edge cubie in the top layer with another cubie on the top layer completely unchanged. (Note that we do not need to remain the structure of the other two layers, as this is exactly what commutators do in finding a nice macro – it undoes irrelevant cubies' position.) In fact, the permutation M is relatively easy to find with no additional tools used, we will exemplify this idea later on. Assuming we have already known what exactly M is, we can apply M, flipping only one edge cubie on the top layer. then we will rotate the top layer to put a different cubie in that cubicle where we originally conducted the flip, then apply M^{-1}, this will flip a new edge cubie while undoes the damage that has been done when we applied M in the other two layers, and we are done [2].

Finding M: There are lots of intuitive and direct ways of finding M, this paper would solely introduce one of them to show how comprehensible this process is. Our objective is to flip UF cubie while leaving the rest of top layer U unchanged. Firstly, we may want to isolate UR cubie from the all the other cubies from U. And this is quite simple, we just need to conduct Rl to firstly separate six upper-left and upper-right cubies from UF, this will be helpful since this permutation would allow rotation of front layer, hence we could now conduct FF to rotate the front layer for 180 degrees, now I want to put the original six upper-left and right cubies into their original position so that we can allow the rotation on the bottom layer. , hence conducting the reverse of Rl, Lr (Note that this is also the conjugated structure, (Rl) (FF) $(Rl)^{-1}$, it functions in putting the UF cubie in DF position, while remaining the indifference structure of the top two layers.) Now we can observe that the original UF cubie is now in DF position, which we should ultimately put it back to UR position but not with direct move such as the inverse of all set of moves above, as this will not change the orientation of the cubie, hence a different returned pathway must be think out, where we firstly conduct d to rotate the down layer 90 degrees clockwise in order to put the original cubie in DL position and eventually bring it back with LrF, and we are done. In conclusion, the complete display of M will be $M = RlFFLrdRlFLr$

Hence by following the previous strategy as we have explained, it is easy to find the macro 'Flip UF, UL', where after applying M, we will just conduct U to rotate the top layer 90 degrees anticlockwise so that the original UL cubie is now in the position of UF cubicle, then applies M^{-1} will undoes all of the damages while flipping UL cubie. Lastly, after flipping UL cubie, we will rotate the top layer back to its initial configuration by u to put UF and UL cubies that are now been flipped into their original position, and the entire procedure has been found by:

$$RlFFLrdRlFLruRlf\ LrDRlff\ LrU$$

Now we will try to find the macro for moving three cubies in a cycle such as

$$(URB,\ UFR,\ ULF)$$

The idea is similar to the previous example, where we will first try to find a macro that can switch two adjacent corner cubies in the top layers while all the other edge cubies remain in its original position, denote the macro Y. We apply Y, then rotate a quarter turn of the top face, then undo the process and we are done, since the process is similar to the permutation

$$(a,b)(b,c)=(a,c,b).$$

Again, to find Y, we do not need additional tools, it is $Y = LrDRdl$, which has the inverse of $lRdrDL$, hence the macro of $(URB,\ UFR,\ ULF)$ is simply $LrDRdlULDrdRlu$.

These are simple demonstration of how commutators are beneficial in finding useful macros. In fact, with enough patience, everyone can solve the cube by 5 fundamental macros, each could be constructed by using similar ideas of the commutators mentioned above.

1. Flip two particular edge cubies
2. Rotate two particle corner cubies
3. Cycle three edge cubies
4. Cycle three corner cubies
5. Swap two corner cubies and two edge cubies

In order to understand the functions of these five macros, we need to acknowledge that these macros only exist under parity constraints. For the first two macros, they are responsible in fixing the orientations of edge and corner cubies, while ensuring that the total rotations always remain a multiple of 360 degrees, in other words, $\sum \text{flip} \equiv 0(\text{mod}2)$.

For the third and fourth macros, they are in charge of generating half of the permutations of the Rubik's cube solvable configurations since three-cycle can generate all the elements of both A_{12} and A_8, here is the proof.

Proposition 3. 1. 1: Alternating group A_n could be generated by three-cycle when $n \geq 3$ [4].

Proof: If two transpositions share an element, their product is a three-cycle, for example

$$(ab)(ac)=(acb).$$

Therefore, for two disjoints transpositions (ab) and (cd), we can express them as

$$(ab)(cd)=(acb)(acd),$$

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since alternating group consists of an even number of transpositions, we can pair them up while expressing each of them by two three-cycle, hence proved.

The fifth macro is also necessary, as it serves to provide odd permutation based on the third and fourth macros, while remaining the parity of the entire cube's permutation even as well.

2. Algorithms evaluation and comparison

One of the reasons about why it is so hard to solve the Rubric's cube (for new player) is the difficulty of preserve—it is hard to move a piece to correct position without break other correct pieces.

CFOP (Cross, F2L, OLL, PLL), also known as the Fridrich Method, is currently the most popular speed-cubing method. It was popularized by Jessica Fridrich in the 1980s and has gradually become a favorable solution for world record runners. The core idea of CFOP is phased optimization, restoring layer-by-layer, and improving the overall reduction efficiency.

CFOP consists of four main steps:

1. Cross -Solve a cross on one face (typically white).
2. F2L (first two layers) -Solve the first two layers by pairing and inserting pairs of corner edges.
3. OLL (Orientation of the Last Layer) -Orient the last layer pieces so all the top colors face up.
4. PLL (Permutation of the Last Layer) -Permute the last layer pieces to their correct positions.

ROUX, developed by Gilles Roux in the early 2000s, the Roux method uses block-building technique that emphasizes move efficiency and minimal rotations. It is renowned for its low overall move count and is especially popular among solvers who excel in one-handed and Fewest Move Challenge (FMC) events. Unlike CFOP, which is highly algorithm-dependent, Roux relies more on the intuition of the player and spatial awareness, making it both an elegant but challenging method.

ROUX consists of four main steps:

1. Build the First Block—a 1×2×3 block on one side of the cube (usually on the left side).
2. Build the Second Block-a second 1×2×3 block on the opposite side of the cube (typically the right side); mirror the strategy used in the first block to solve the opposite side.
3. Solve the Corners of the Last Layer (CMLL)
4. Solve the remaining 6 edges and 4 center.

The first two steps are performed intuitively.

Corner First (CF) is the oldest solving method, which is also how Rubik used to solve the cube. It is a pretty nice and fun method to learn. However, it requires much more time on edges observation, making the restoring progress stuck usually.

CF consists five main steps:

1. Solving the corners like a 2×2×2 Rubik's cube
2. Rotate the middle slice to align center.
3. Solve the middle slice
4. Solve the position of the top and bottom slice
5. Solve the direction of the top and bottom slice

Overall, CFOP, ROUX and CT are three distinct restoring method as their unique way to breaking down the process. CFOP has a really clear solving idea—complete one layer by one layer—each step has a specific purpose. However, the drawback of it is the difficulties on preserving the layer that have been restored; we have to consider the consequence of every formula that we created and add a lot of stabilizers over the formula. Thus, CFOP requires reach up to 78 distinct algorithms and 55-60 moves to complete the cube. In general, CFOP's formulas are usually a big trouble for new player to mesmerize and utilize them. For the ROUX method, it is similar to CFOP but require

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less steps in average-require less rotations. ROUX deal with the cube region-to-region. Due to its unique reduction steps, it shows big advantage on one-handed reduction competition. However, it requires player's strong spatial imagination ability and path planning ability. Additionally, compare with other strategies, ROUX doesn't have so much documents on internet that help players to learn. CF shows a nice and obvious division of the rubric's cube mathematically and mechanically: corner, edge and center. It isolates each type of cubes directly into independent set. As the result, it requires few algorithms to complete each stage. The only challenge for this method is about finding suitable methods to restore edge. Certain edge configurations might introduce parity challenges that require additional corrective moves. In some scenarios, solving corners first can lead to a lower overall move count, particularly in optimized or theoretical solving contexts like the Fewest Moves Challenge (FMC). It is also beneficial for machines to understand the group of cubes.

3. Graphing Rubik's cube

For Rubik's cube, there are several methods to restore it and we have been discussed three ways in previous passages. These methods have a different division of the cubies set at each step; thus, we should represent them in different generators. Our goal is to create a visual representation of Rubric's cube.

General speaking, we can pick the rotation of 6 faces—U, D, F, B, L and R—for the generators. Each of them is a permutation of the cubies on one face. Their combination constructs the group of rubric's cubes. It is obvious that the group (orbits) of edge and corner are disjoint due to their mechanical property, so we can first divide Rubik's cube G into two smaller subgroups that share identical generators: edge group and corner group (which is also corresponding to method Corner First's idea). Moreover, the reason why we separate them is to create a better visual effect.

The graph of the edges of a single face (The red one) is a 4-cycle double cyclical group. We also consider the orientation of the edge as a state. So, there are two disjoint cycles.

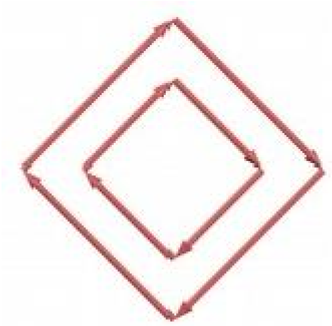

Figure 2. Edge cycle of a face

Similarly to the corner cycle:

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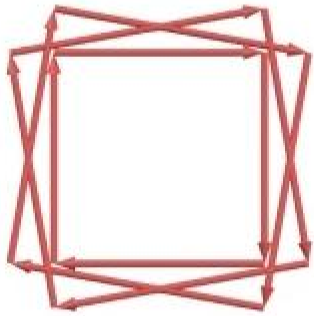

Figure 3. Corner cycle of a face

Combining 6 faces together, we get the whole graph. Edge:

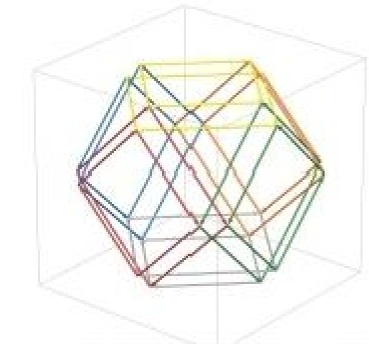

Figure 4. 6 Edge cycle combined together

The corner:

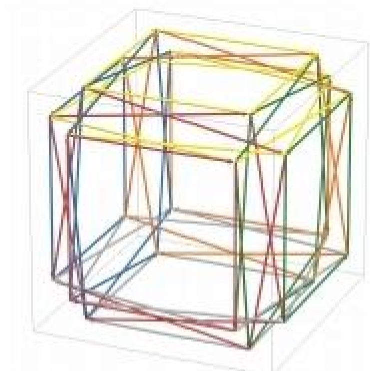

Figure 5. 6 corner cycle combined together

The combination of these two graphs is the graph of 3×3 Rubric's cube group G.
The position group of corners is Isomorphism to A_8while the edge is isomorphism to A_{12}.
Consider the polarity of edges and corners, the possible number of rotations is 3^7 for corner and $2^{11}$for edge. So, we can simply divide Gin this way:

$$G = \left(A_8 \rtimes Z_3^7\right) \times \left(A_{12} \rtimes Z_2^{11}\right)$$

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Where:

A_{12} -permutation of 12 edges

A_8 -permutation of 8 corners

Z_3^7 -direction of 8 corners

Z_2^{11} -direction of 12 edges

Sequence of solving the group of cubes by CF:

$$A_8 - Z_3^7 - A_{12} - Z_2^{11}$$

The first and second stage of CF is identical to the way of solving 2×2×2 rubric's cube.

We're goanna to skip this part.

For the third step, we can restore the position and direction of 7 of the edges on the top and bottom slice, typically yellow and white face, by intuition. The formula required to permute single edge from middle slice to top/bottom slice: $RER'E'R'ER$

For the last step, to solve the position of 4 edges on middle slice, we have this formula: R^2ER^2E'

The formula to change the direction of 2 edges:

$$(UE)^3U^2(M'U)^3$$

Overall, the process of CF to solve rubric's cube group is like this:

$$G = \left(A_8 \rtimes Z_3^7\right) \times \left(A_{12} \rtimes Z_2^{11}\right)$$

$$\left(A_4 \rtimes Z_3^3\right) \times \left(A_{12} \rtimes Z_2^{11}\right)$$

$$A_4 \times \left(A_{12} \rtimes Z_2^{11}\right)$$

$$\left(A_{12} \rtimes Z_2^{11}\right)$$

$$\left(A_4 \rtimes Z_2^3\right)$$

$$Z_2^3$$

$$Z_2$$

$$e$$

ROUX: The third and fourth stage of roux method involves the rotation of top layer and a middle slice. These legal moves are corner-preserved What we should focus on is the permutation of the six unsolved edges. We denote the direction of the edge is correct when it's color of the color of top or bottom layer is facing vertically. The method first adjusts the direction of six edges, therefore when we are solving the position, we can just apply double flip of middle surrounding slices (for instance, M^2, F^2, L^2, R^2 and B^2) so the direction is preserved. The letter a, b, c, d, m, n represent six unsolved positions, their permutation is isomorphism to A_6.

The blue cycle represents the permutation of U, while the red cycle represents the permutation of M, together, we get the graph of MU.

$$M = (acmn)$$

$$U = (abcd)$$

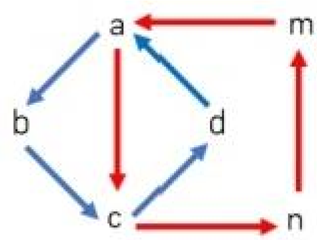

Figure 6. Graph of MU

One M operation changes the direction of 4 edges on the orbit, combining with rotation U, it is enough to solve all six edges' direction.

Formula: M'UMU

For the next stage, since the position of six edges are isomorphism to A6, the combination of U and M are not enough to solve this group since these two operations cannot construct a group (non-commutative). But U and M can solve most situation

Basic Formula: $MU^2M'U^2$

Overall, the process that ROUX decomposes Rubric's cube is:

$$G = \left(A_8 \rtimes Z_3^7\right) \times \left(A_{12} \rtimes Z_2^{11}\right)$$

$$\left(A_4 \rtimes Z_3^3\right) \times \left(A_6 \rtimes Z_2^5\right)$$

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$$A_4 \times (A_6 \rtimes Z_2^5)$$

$$A_6 \rtimes Z_2^5$$

$$A_6$$

$$e$$

Sometimes, the cubies that the formula adjust will not completely corresponding to real situation, the cubies may at different position. We can apply conjugate and commutator on the basic formulas to move the target cubes to corresponding position where the formula could adjust.

Additionally, sometimes the "intuition" part of solving methods is also considerable. Typically, the more "intuition" part in a methods meaning the more process we can restore easily without formula (Though the "intuition" requires a lot of training). We can evaluate the feature of formulas by analyzing the target group which corresponding to each stage and the method do at each stage [5].

4. Evaluation

4.1. Strengths

It must be acknowledged that this marks our inaugural experience in composing such formal academic writing. Through this process, we came to recognize the indispensable role of mentor guidance and scholarly discourse. While substantial content remains foundational, we have indeed contributed original proofs -most notably regarding the parity of edge cubies' orientation, where we devised a coordinate system to rigorously establish conclusions. This constitutes an instructive methodological approach. Furthermore, our work provides fellow high school students with a valuable resource and methodological framework for studying group theory, particularly in bridging abstract concepts with tangible combinatorial applications.

4.2. Weaknesses

We faced many challenges while writing this paper. At first, we wanted to study interesting topics like "God's Number" (the fewest moves needed to solve any Rubik's Cube). For example, we tried to find the "hardest" cube configuration to calculate this number. But since none of us were good at programming, we had to pick a different topic. Another big question we kept asking was: "What's the smallest number of standard moves needed to solve the cube?" We couldn't answer this either, because our group theory knowledge was still limited. These struggles taught us that good math research needs both creative ideas and strong technical skills. We'll keep learning group theory and come back to solve these puzzles someday!

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5. Conclusion and acknowledgement

Overall, in this paper, we introduced some basic ideas and properties of Rubik's cube by relating to group theory, while introducing the main functions of commutator and conjugation, and eventually proved that five macros are sufficient in solving the Rubik's cube. We also analyzed and compared the process of three restoring methods by depicting the graphs of group.

With this acknowledgment, we extend our deepest gratitude to all mentors and peers who contributed to the completion of this thesis. we are profoundly indebted to Professor Dan Ciubotaru for your visionary academic guidance, strategic oversight of the research direction, and generous sharing of expertise. Your meticulous revisions and intellectual rigor have deepened our understanding of mathematical inquiry. Special recognition is owed to Teaching Assistant Qiu for your selfless dedication in course instruction and post-lecture consultations. Your patient elucidation of formula derivations, insightful suggestions for structural optimization, and timely academic support even during late-night hours forms the cornerstone of this work. This scholarly journey has illuminated that academic cultivation demands both extensive contemplation and distilled wisdom, but above all, enlightened mentorship. To both of you, we offer our utmost respect and sincere appreciation.

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